## 8.5 Advanced Confidence Interval

### Confidence Intervals

**Def:** Let $\mathbf{X}$ be a random sample from a distribution with parameter $\theta$. Let $g(\theta)$ be a real-valued function of $\theta$. Let $A \leq B$ be two statistics with the property that for all values of $\theta$,

$$P(A < g(\theta) < B) \geq \gamma$$

Then the random interval $(A, B)$ is called a confidence $\gamma$ confidence interval for $g(\theta)$. If the inequality for $\gamma$ is actually an equality, then the confidence interval is called **exact**. After the values of the random sample $\mathbf{X} = \mathbf{x}$ have been observed, and the values $A = a$ and $B = b$ are computed, the interval $(a, b)$ is called the **observed value** of the confidence interval.

**Thm:** Let $\mathbf{X}$ be a random sample from $N(\mu, \sigma^2)$. Consider estimators $\bar{X}$ and $S$ for $\mu$ and $\sigma$:

$$\bar{X} = \frac{1}{n}\sum X_i \qquad S = \sqrt{\frac{1}{n-1}\sum (X_i - \bar{X})^2}$$

For each $0 < \gamma < 1$, then interval $(A, B)$ with the following endpoints is an exact confidence interval for $\mu$:

$$A = \bar{X} - F_{n-1}^{-1}\left(\frac{1+\gamma}{2}\right)\frac{S}{\sqrt{n}} \qquad B = \bar{X} + F_{n-1}^{-1}\left(\frac{1+\gamma}{2}\right)\frac{S}{\sqrt{n}}$$

where $F_{n-1}^{-1}$ is the quantile function for the $t$-distribution with $n-1$ degrees of freedom.

**Ex 1:** A batch of stout beer is best when it has an original gravity (OG) close to 1.071. The particular OG of a batch depends on a number factors (like temperature, rest time, recipe, etc.) but is (approximately) Normally distributed. Suppose we sample 5 OG measurements from a batch of beer:

$$1.067 \quad 1.060 \quad 1.077 \quad 1.072 \quad 1.067 \qquad \text{with } \bar{x} = 1.0686 \text{ and } s = 0.0064$$

Using this sample of 5 OG measurements from a batch of beer, construct a 95% confidence interval for $\mu$.

### One-sided Confidence Intervals

For any given confidence level $\gamma$, it is possible to construct infinitely many confidence intervals $(A, B)$ so that $P(A < \mu < B) = \gamma$. However, among all such intervals, the symmetric interval has the shortest length.

But there are cases we may be interested in an asymmetric interval.

**Def:** A **one-sided $\gamma$ confidence interval** for $g(\theta)$ is a random interval of the form $(A, \infty)$ of $(-\infty, B)$ so that

$$P(A < g(\theta)) \geq \gamma \qquad \text{or} \qquad P(g(\theta) < B) \geq \gamma$$

That is, one-sided confidence intervals provide lower or upper bounds (but not both) for the parameter.

**Ex 2:** Using the sample of 5 OG measurements from a batch of beer, construct a 90% lower confidence interval for $\mu$.

### Confidence Intervals for General Parameters

When we constructed a confidence interval for $\mu$ for sample data with $X_i \sim N(\mu, \sigma^2)$, the first step was to consider the random variable

$$T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

which has the $t$-distribution with $n-1$ degrees of freedom. In particular, we note that while the distribution of $\bar{X}$ depends on both $\mu$ and $\sigma^2$, and while $\mu$ is used in the formula to calculate the value of $T$, the distribution of $T$ itself depends on neither $\mu$ nor $\sigma^2$. Such a random variable is called a Pivotal Quantity for $\mu$.

**Def:** Suppose $\mathbf{X}$ is a random sample from a distribution that depends on one or more parameters $\theta$. Let $T(\mathbf{X}, \theta)$ be a random variable whose *distribution* is the same for all values of $\theta$. Then $T$ is called a **pivotal quantity** for $\theta$.

In order to effectively use the pivotal in order to construct a confidence interval for $\theta$, we need a way to invert (or solve) the pivotal for the parameter.

Consider again the procedure for finding a confidence interval for $\mu$. We started with

$$\gamma = P(-c_\gamma < T < c_\gamma)$$

using the fact that the distribution of $T$ did not depend on $\mu, \theta$. Then, we applied algebraic maneuvers to each term in the inequalities in order to isolate $\mu$:

$$\gamma = P(-c_\gamma < T < c_\gamma) = P\left(-c_\gamma < \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} < c_\gamma\right) = P\left(\bar{X} - c_\gamma \frac{S}{\sqrt{n}} < \mu < \bar{X} + c_\gamma \frac{S}{\sqrt{n}}\right)$$

Suppose $r(t, \mathbf{x})$ is a function representing these algebraic operations. In particular,

$$r(t, \mathbf{x}) = \left(t \cdot \frac{S}{\sqrt{n}} - \bar{X}\right) \cdot (-1) = \bar{X} - t \cdot \frac{S}{\sqrt{n}}$$

and $r(T, \mathbf{x}) = \mu$.

Note that $r$ is a *decreasing* function of $t$. Then

$$\gamma = P(-c_\gamma < T < c_\gamma) = P\Big(r(-c_\gamma, \mathbf{x}) > r(T, \mathbf{x}) > r(c_\gamma, \mathbf{x})\Big) = P\left(\bar{X} - c_\gamma \frac{S}{\sqrt{n}} < \mu < \bar{X} + c_\gamma \frac{S}{\sqrt{n}}\right)$$

We are going to mimic this procedure, but now for a general parameter $\theta$ and general pivotal quantity $T(\mathbf{X}, \theta)$.

**Thm:** Let $r(t, \mathbf{x})$ be a function of a number $t$ and the data $\mathbf{x}$ so that $r(T(\mathbf{x}, \theta), \mathbf{x}) = \theta$. Let $G^{-1}$ be the quantile function for the pivotal quantitty $T$. Moreover, assume that $r$ is a **decreasing** function of $t$. Let $0 < \gamma < 1$ and let $\gamma_1 < \gamma_2$ with $\gamma = \gamma_2 - \gamma_1$. Consider statistics $A(\mathbf{X})$ and $B(\mathbf{X})$ given by

$$A = r\left(G^{-1}(\gamma_2), \mathbf{X}\right) \qquad B = r\left(G^{-1}(\gamma_1), \mathbf{X}\right)$$

Then $(A, B)$ is an *exact* confidence interval for $\mu$.

**Note:** If $r$ is a strictly increasing function of $t$, then the role of $\gamma_1$ and $\gamma_2$ above are reversed.

*Proof.* We just prove the case where $r$ is decreasing. The other case is given in the textbook. Let $c_1 = G^{-1}(\gamma_1)$ and $c_2 = G^{-1}(\gamma_2)$. Then

$$\gamma = \gamma_2 - \gamma_1 = P(T < c_2) - P(T \leq c_1) = P(c_1 < T < c_2) = P(r(c_1, \mathbf{x}) > r(T, \mathbf{x}) > r(c_2, \mathbf{x}))$$

where crucially, we used the fact that since $r$ is a decreasing function $t$, then it reverses the direction of inequalities. $\square$

We conclude with an example for building a confidence interval for the population variance $\sigma^2$.

**Ex 3:** Suppose $X_1, \ldots, X_n$ is a sample with $X_i \sim N(\mu, \sigma^2)$ with $\mu$ and $\sigma^2$ unknown. Recall that the MLE for $\sigma^2$ is

$$\hat{\sigma}^2 = \frac{1}{n} \sum (X_i - \bar{X})^2$$

To build a confidence interval for $\sigma^2$ using $\hat{\sigma}^2$, we first need to find a pivotal quantity for $\hat{\sigma}^2$. Think back to when we first discussed this estimator. Are there any variables related to $\hat{\sigma}^2$ that have the same distribution regardless of the value of $\mu$ and $\sigma^2$?

In fact, it turns out the the variable $T = \frac{n}{\sigma^2}\hat{\sigma}^2 \sim \chi^2(n-1)$, which does not depend on either unknown parameter! This shows that $T$ is a pivotal quantity.

Now, we need to find a way to invert this pivotal; that is, we need to solve $t = \frac{n}{\sigma^2}\hat{\sigma}^2$ for $\sigma^2$. In this case,

$$t = \frac{n}{\sigma^2}\hat{\sigma}^2 \Longleftrightarrow \sigma^2 = \frac{n\hat{\sigma}^2}{t}$$

So our function $r$ is

$$r(t, \mathbf{x}) = \frac{n\hat{\sigma}^2}{t}$$

Note that we can write the formula for this function using just the input $t$ and functions of the data $\mathbf{X}$. Note also that $r$ is a *decreasing* function of $t$.

Let $G^{-1}$ be the quantile function for $\chi^2(n-1)$; there isn't a closed form for this function, so we'll leave it as $G^{-1}$ in our formula. Then our confidence interval $(A, B)$ is given by

$$A = r\left(G^{-1}(\gamma_2)\right) = \frac{n\hat{\sigma}^2}{G^{-1}(\gamma_2)} \qquad B = r\left(G^{-1}(\gamma_1)\right) = \frac{n\hat{\sigma}^2}{G^{-1}(\gamma_1)}$$

**<u>Ex 4:</u>** Using the sample of 5 OG measurements from a batch of beer, construct a 95% confidence interval for $\sigma^2$.