Homework 9: 4/10 - 4/14

STA 336 Due 11:59pm Wednesday, April 19 Name:

Instructions: Write-up complete solutions to the following problems and submit answers on Gradescope. Your solutions should be neatly-written, show all work and computations, include figures or graphs where appropriate, and include some written explanation of your method or process (enough that I can understand your reasoning without having to guess or make assumptions). A rubric for homework problems appears on the final page of this assignment.

- Unless otherwise noted, problem numbers are taken from the 4th edition of DeGroot and Schervish's *Probability and Statistics.*

Monday 4/10

Section 9.1: 9, 13, 14

Wednesday 4/12

Section 9.1: (8, 10), 15 [problems 8 and 10 will be graded as a single problem]

Additional Problems

AP1. In class, we defined the p-values of a sample as follows:

Def: Let $\{\delta_c\}$ be a collection of hypothesis tests where δ_c has size α_c, such that for $\alpha_c < \alpha_{c'}$, if δ_c rejects H_0 when $\mathbf{x}$ is observed, then $\delta_{c'}$ also rejects H_0 when $\mathbf{x}$ is observed. The **p-value** for the observed sample $\mathbf{x}$ is the smallest level α_0 such that $\mathbf{x}$ is in the rejection region for δ_c.

Consider a simple null hypothesis $H_0 : \theta = \theta_0$ with alternative hypothesis $H_1 : \theta \neq \theta_0$. Let $T(\mathbf{X})$ be a test statistic and consider a collection of tests $\{\delta_c\}$ of the form δ_c: "Reject H_0 if $T \geq c$".

(a) Show that the collection $\{\delta_c\}$ has the property that for $\alpha_c < \alpha_{c'}$, if δ_c rejects H_0 when $\mathbf{x}$ is observed, then $\delta_{c'}$ also rejects H_0 when $\mathbf{x}$ is observed. *Hint: If $\alpha_c < \alpha_{c'}$ what must be true about c and c'?*

(b) Suppose $T(\mathbf{x}) = t$. Show that the p-value for this observed sample is

$$p\text{-value} = P(T \geq t|\theta_0).$$

Friday 4/7

Section 9.1: 16, ~~17~~ (Problem 9.1.17 was originally listed in the homework, but has since been removed.)

Additional Problems

AP2. Suppose $X_1, \ldots, X_n$ are iid $N(\mu, 1)$ with $\mu > 0$ unknown. We are interested in testing the following hypotheses:

$$H_0 : \mu = 0 \qquad H_1 : \mu \neq 0$$

In the exercise, you will construct a likelihood ratio test for these hypotheses.

(a) Find the formula for the likelihood function $f(\mathbf{x}|\mu)$.

(b) Recall that the likelihood ratio statistic is defined as

$$\Lambda(\mathbf{X}) = \frac{\sup_{\mu \in \Omega_0} f(\mathbf{x}|\mu)}{\sup_{\mu \in \Omega} f(\mathbf{x}|\mu)}$$

Find formulas for the numerator and the denominator of the likelihood ratio statistic. *Hint:* To compute the denominator of the likelihood ratio statistic, recall that the MLE for μ is $\bar{X}$.

(c) Show that the likelihood ratio statistic $\Lambda(\mathbf{x})$ can be simplified as

$$\Lambda(\mathbf{x}) = \exp\left(-\frac{n(\bar{x})^2}{2}\right)$$

Hint: Use the oft-cited identity:

$$\sum_{i=1}^{n}(x_i - \mu)^2 = n(\bar{x} - \mu)^2 + \sum_{i=1}^{n}(x_i - \bar{x})^2$$

(d) Recall that a likelihood ratio test is of the form: "Reject H_0 if $\Lambda(\mathbf{x}) \leq c$," where c is a constant with $0 < c < 1$. Show that the rejection region for a likelihood ratio test can also be expressed as

$$S_1 = \left\{\mathbf{x} : |\bar{x}| \geq \sqrt{\frac{-2\log c}{n}}\right\}$$

In other words, a likelihood ratio test is of the form "Reject H_0 if $\bar{x}$ is far from 0."

(e) Find a value of c so that the likelihood ratio test has size $\alpha = 0.05$. *Hint: If $Z \sim N(0,1)$, for what value of z is $P(|Z| \geq z) = 0.05$?*

General Rubric

Points	Criteria
5	The solution is correct **and** well-written. The author leaves no doubt as to why the solution is valid.
4.5	The solution is well-written, and is correct except for some minor arithmetic or calculation mistake.
4	The solution is technically correct, but author has omitted some key justification for why the solution is valid. Alternatively, the solution is well-written, but is missing a small, but essential component.
3	The solution is well-written, but either overlooks a significant component of the problem or makes a significant mistake. Alternatively, in a multi-part problem, a majority of the solutions are correct and well-written, but one part is missing or is significantly incorrect
2	The solution is either correct but not adequately written, or it is adequately written but overlooks a significant component of the problem or makes a significant mistake.
1	The solution is rudimentary, but contains some relevant ideas. Alternatively, the solution briefly indicates the correct answer, but provides no further justification
0	Either the solution is missing entirely, or the author makes no non-trivial progress toward a solution (i.e. just writes the statement of the problem and/or restates given information)
Notes:	For problems with multiple parts, the score represents a holistic review of the entire problem. Additionally, half-points may be used if the solution falls between two point values above.