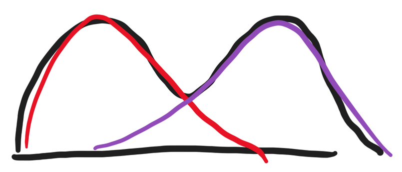
Method moments - oldest technique

Distribution of ratios forehead to body length



5 parameters: 2 means, 2 variances, 1 ratio of population sizes

Pearson computed 1st 5 sample moments of data
compared to 1st 5 theoretical moments
Get 5 equations, 5 unknowns
Solve. Solution will be estimates for unknown parameters.

Def/ If X is a r.v. the k^{th} moment of X μ_k is $E[X^k]$.

Ex/ Suppose $X \sim \text{Beta}(\alpha,\beta)$. Compute $E[X^2]$.

$$E[X^2] = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \overbrace{\int \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1}\,dx}^{\text{Beta}(\alpha+2,\beta)} = 1$$

$$\boxed{= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}} \quad \text{using} \quad \Gamma(x+1) = \Gamma(x)\cdot x$$

Let $X_1, X_2, \dots, X_n$ be a random sample. The k^{th} sample moment M_k as $\frac{1}{n}\sum_{i=1}^{n} X_i^k$. Note: 1st sample moment is sample mean $M_1 = \bar{X}$

Note: Sample moments are statistics.

Thm The k^{th} sample moment is a consistent and unbiased estimator for the k^{th} moment of the distribution.

Proof/ By Weak Law of Large Numbers,

$$\frac{1}{n}(X_1^k + X_2^k + \dots + X_n^k) \xrightarrow[n\to\infty]{} E[X_i^k] = \mu_k$$

$$E[M_k] = E\left[\frac{1}{n}\sum X_i^k\right] = \frac{1}{n}\sum E[X_i^k] = \frac{1}{n}\sum \mu_k = \mu_k$$

Caution: Not every estimator built from sample moments is unbiased.

Recall sample variance

$$\hat{\sigma}^2 = \frac{1}{n}\sum (X_i - \bar{X})^2$$

is biased estimator of σ^2 if $X_i \sim N(\mu, \sigma^2)$.

Def/ The method of moments. To estimate k parameters of a distribution, express 1st k moments of dist. in terms of those parameters, solve system of equations for parameters. Replace all theoretical moments with sample moments. This will give collection of estimators for parameters.

Ex 1 Suppose $X_1, X_2, \dots, X_n \sim \text{Expo}(\theta)$ with θ unknown.
Find MoM estimator for θ.

$$E[X_1] = \frac{1}{\theta} \implies \theta = \frac{1}{E[X_1]}$$

$$\hat{\theta} = \frac{1}{\bar{X}} \quad \text{by replacing } E[X] \text{ with } \bar{X}.$$

Note: This is also the MLE.

MoM is not always MLE

Ex 2 Suppose $X_1, X_2, \dots, X_n \sim \text{Beta}(\alpha,\beta)$. Find MoM for (α,β).

$$\mu_1 = E[X_1] = \frac{\alpha}{\alpha+\beta} \qquad \mu_2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Solve these equations for α, β.
Hint: $\alpha+\beta = \frac{\alpha}{\mu_1}$ from Eq. 1

After algebra

$$\alpha = \frac{\mu_1(\mu_1-\mu_2)}{\mu_2-\mu_1^2} \qquad \beta = \frac{(1-\mu_1)(\mu_1-\mu_2)}{\mu_2-\mu_1^2}$$

Now replace with sample moments

$$\hat{\alpha} = \frac{M_1(M_1-M_2)}{M_2-M_1^2} \qquad \hat{\beta} = \frac{(1-M_1)(M_1-M_2)}{M_2-M_1^2}$$

Note: no closed form expression for MLE