Probability

Prof. Wells

STA 209, 4/7/23

Outline

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- Introduce key definitions for probability theory
- Define conditional probability
- Investigate Bayes Rule for conditional probabilities

Section 1

Probability Theory

Random Processes and Events

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    - For brevity, we'll often denote events using capital letters like A, B, C.
    - For example, we might say: Let A be the event "the sum of values on two dice equals 7"

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- Since probabilities are defined in terms of proportions, they will always be values between 0 and 1.
- For brevity, we'll represent statements like *the probability of the event "the coin lands heads" is 50%* using the notation:

$$P(\text{the coin lands heads}) = 0.5 \quad \text{or} \quad P(\text{Heads}) = 0.5 \quad \text{or} \quad P(H) = 0.5$$

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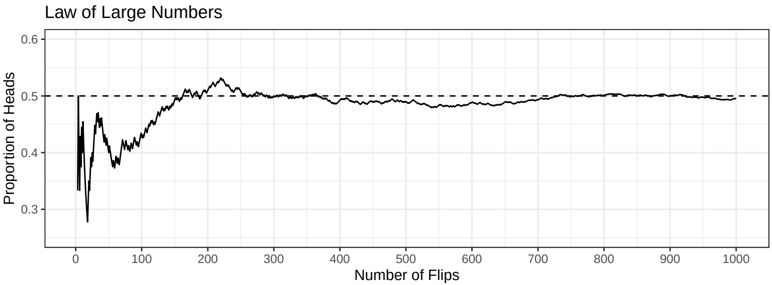

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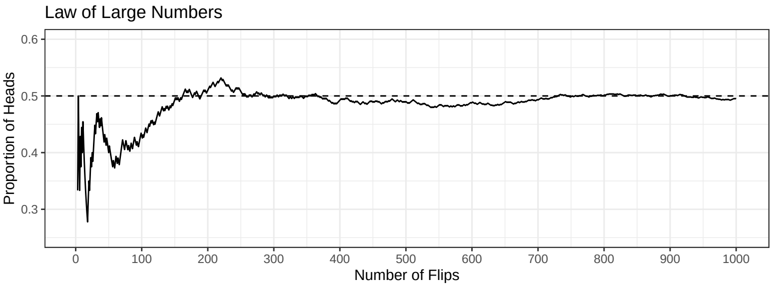

- Note that the proportion of heads deviates (significantly) from 0.5 during the first 50 flips, but stabilizes around 0.5 by 1000 flips.

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- Whenever we discuss probability, we are always (either explicitly or implicitly) defining a probability model.
- A large component of statistical inference is comparing observed data to the results we would expect under a certain probability model.

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$$P(\text{ starts with f }) = P(\text{ roll a 4 or roll a 5 }) = P(\text{ roll a 4 }) + P(\text{ roll a 5 }) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

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$$P(\text{ roll something other than a 1 }) = 1 - P(\text{ roll a 1}) = 1 - \frac{1}{6} = \frac{5}{6}$$

Section 2

Conditional Probability

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- **Note!** Saying that two events are *disjoint* is not the same as saying they are independent.
  - Disjoint events cannot simultaneously occur; while for independent events, knowing that one occurs gives no information about whether the other occurs.
  - If two events are disjoint, and if you know one has occurred, then you automatically know the other cannot occur. Disjoint events are as far away from independence as possible!

Multiplication Rule for Independent Events

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  - Moreover, since we have a fair coin, then $P(A) = P(B) = \frac{1}{2}$ and so

$$P(A \text{ and } B) = P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

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$$\begin{aligned} P(A \text{ not in bootstrap}) =& P(A \text{ not 1st}) \cdot P(A \text{ not 2nd}) \cdot P(A \text{ not 3rd}) \cdot P(A \text{ not 4th}) \\ =& \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{81}{256} \approx 0.32 \end{aligned}$$

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  - The event A occurs if we get one of HT or HH, so $P(A) = \frac{1}{2}$.

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To relate $P(A|B)$ and $P(B|A)$, we use the following theorem:

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