Probability and Random Variables

Prof. Wells

STA 209, 4/10/23

Outline

In this lecture, we will. . .

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In this lecture, we will...

- Investigate Bayes Rule for conditional probabilities
- Define and investigate random variables
- Compute the mean and standard deviation of random variables

Section 1

Conditional Probability

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- The **conditional probability** of an event A given another event B is
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Theorem (General Multiplication Rule)

For any events A and B,

$$P(A \ and \ B) = P(A|B)P(B) = P(B|A)P(A)$$

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  - Disjoint events cannot simultaneously occur; while for independent events, knowing that one occurs gives no information about whether the other occurs.
  - If two events are disjoint, and if you know one has occurred, then you automatically know the other cannot occur. Disjoint events are as far away from independence as possible!

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  - Let A be the event that the first flip is heads and B be the event that the second is heads.
  - Since the result of the first flip has no bearing on the second flip, then A and B are independent.
  - Moreover, since we have a fair coin, then $P(A) = P(B) = \frac{1}{2}$ and so

$$P(A \text{ and } B) = P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Practice with Probability

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$$\begin{aligned} P(A \text{ not in bootstrap}) =& P(A \text{ not 1st}) \cdot P(A \text{ not 2nd}) \cdot P(A \text{ not 3rd}) \cdot P(A \text{ not 4th}) \\ =& \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{81}{256} \approx 0.32 \end{aligned}$$

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  - What is $P(\text{ car is red}|\text{ car is Ferrari})$?
  - But what is $P(\text{ car is Ferrari}|\text{ car is red})$?

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To relate $P(A|B)$ and $P(B|A)$, we use the following theorem:

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Let A and B be events. Then

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- In general, the value of ratio $\frac{P(H_0 \text{ is true})}{P(\text{ extreme statistic })}$ is difficult to calculate, and requires significant prior knowledge about the data generation process.

Section 2

Random Variables

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- We use equations to express events associated to random variables.
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- Events associated to variables have probabilities of occurring.
  - $P(X = 5) = .5$ means X has 50% probability of taking the value 5.

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- Some discrete variables can be well-described by continuous variables:
  - The height of a random person selected from a large population.
  - The proportion of heads in a long sequence of coin flips.

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- Playing the casino game is very similar to drawing a random coin from the purse.

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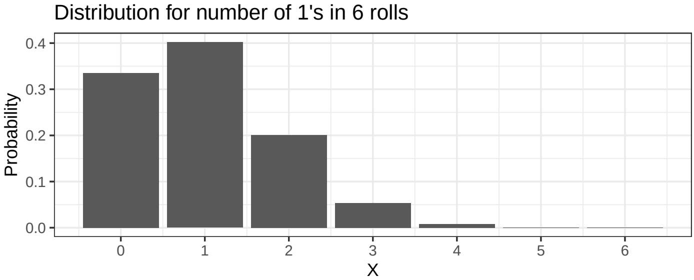

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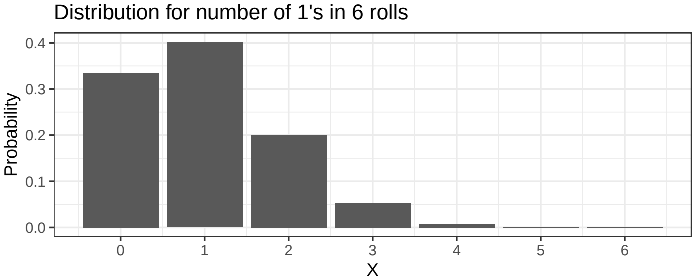

- Heights of bars are probabilities
  - This is analogous to rescaling a histogram to have heights equal to proportions, rather than counts

Expected Value

The **expected value** (or mean) of a discrete random variable X is

$$E[X] = x_1 P(X = x_1) + x_2 P(X = x_2) + \cdots + x_n P(X = x_n) = \sum_{i=1}^{n} x_i P(X = x_i)$$

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$$E[X] = x_1 P(X = x_1) + x_2 P(X = x_2) + \cdots + x_n P(X = x_n) = \sum_{i=1}^{n} x_i P(X = x_i)$$

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- But also notice that

$$E[X] = \frac{1}{10}(2 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 5) = \frac{1 + 1 + 2 + 2 + 2 + 2 + 3 + 4 + 5 + 5}{10}$$

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- The expected value of a random variable is the arithmetic mean of a data set, where each observation in the data occurs with frequency equal to its probability.

The Law of Large Numbers, again

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This is a generalization:

Theorem (The Law of Large Numbers)

Let X be a random variable whose value depends on a random experiment. Suppose the experiment is repeated n times and let $\bar{x}_n$ denote the arithmetic mean of the values of X in each trial. As n gets larger, the arithmetic mean $\bar{x}_n$ approaches the expected value $E[X]$ of that variable.

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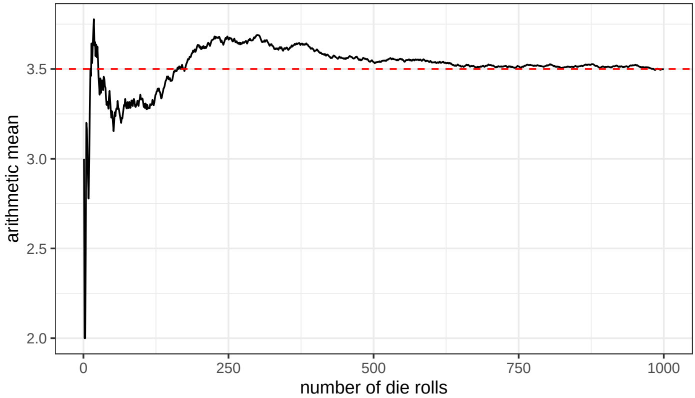

Variance and Standard Deviation

The **variance** of a discrete random variable X with mean μ is

$$\begin{aligned}\mathrm{Var}(X) =&(x_1 - \mu)^2 P(X = x_1) + (x_2 - \mu)^2 P(X = x_2) + \cdots + (x_n - \mu)^2 P(X = x_n) \\ =&\sum_{i=1}^{n} (x_i - \mu)^2 P(X = x_i)\end{aligned}$$

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- Show that the standard deviation of a variable X which takes value 1 with probability p and 0 with probability $1-p$ is

$$\mathrm{SD}(X) = \sqrt{p(1-p)}$$